ON THE GEOMETRY OF CANONICAL CURVES OF ODD GENUS

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Dedicated to Robin Hartshorne on his 60th birthday

Introduction

Fix an algebrically closed field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) = 0$. Let C be a projective nonsingular curve of genus 5 defined over $\mathbf{k}$, and let $W^1_4 \subset \mathrm{Pic}^4(C)$ be the subscheme of divisor classes of degree 4 and dimension 1. W^1_4 is a curve, which is irreducible and nonsingular of genus 11 if C is general, and can be identified with the singular locus of the theta divisor $W_4 \subset \mathrm{Pic}^4(C)$. Under the Gauss rational map W^1_4 is mapped 2–1 onto a curve $\Gamma \subset \mathbb{P}^4$ which is nonsingular of degree 10 and genus 6. Γ is the set of vertices of the rank four quadrics containing the canonical image $\kappa(C) \subset \mathbb{P}^4$. The planes contained in the quadric Q_L, $L \in W^1_4$, form the congruence (2-dimensional family) of 4-secant planes to $\kappa(C)$. On each plane π of the congruence we have the set $(\kappa(C) \cup \Gamma) \cap \pi$ consisting of 5 points. For a general choice of π these points are distinct and contained in a unique conic $F \subset \pi$ which can be obtained as the locus of first order foci of the congruence, whereas the points $\big(\kappa(C) \cup \Gamma\big) \cap \pi$ are the second order foci (see below for the definitions of first and second order foci).

This geometrical configuration can be viewed as the first case of a whole series in two different ways. Firstly we can consider a sufficiently general curve C of genus $g \geq 5$, and the locus $W^1_{g-1} \subset \mathrm{Pic}^{g-1}(C)$, which can be identified with the singular locus of the theta divisor W_{g-1}. W^1_{g-1} has pure dimension $g-4$; the projectivized tangent space at a point $L \in W^1_{g-1} \backslash W^2_{g-1}$ is the linear space $v_L \subset \mathbb{P}^{g-1}$ of dimension $g-5$ vertex of the quadric Q_L of rank four containing the canonical curve $\kappa(C)$, which is the projectivized tangent cone to W_{g-1} at L.

Each quadric Q_L has two rulings of $\mathbb{P}^{g-3}$'s, and when L varies in $W^1_{g-1} \backslash W^2_{g-1}$ they form a family of dimension $g-3$. On each sufficiently general $\mathbb{P}^{g-3}$ of the family there is a rational normal curve of first order foci. It has been shown in [CS] that the curve C can be recovered from this family, and this has been used to give a proof of Torelli's theorem.

Another extension of the genus 5 configuration can be introduced naturally, and it is the object of the present paper. Let's consider, for any odd $g = 2n+1$, $n \geq 2$, a sufficiently general projective irreducible nonsingular curve C of genus g, and the locus $W^1_{n+2} \subset \mathrm{Pic}^{n+2}(C)$. By Brill-Noether theory it is known that W^1_{n+2} is a nonsingular irreducible curve, whose genus has been computed by Kempf [K2] and by Pirola [P] ($g = 11$ for $n = 2$). For each $L \in W^1_{n+2}$ the projectivized tangent cone to W_{n+2} at L is a variety $X_L \subset \mathbb{P}^{2n}$, of degree n and dimension $n+1$, and

$$X_L = \bigcup_{D \in |L|} \langle D \rangle$$

i.e. X_L is described as a 1-dimensional family of n-spaces, $(n+2)$-secant to $\kappa(C)$. The vertex v_L of X_L varies in a curve $\Gamma \subset \mathbb{P}^{2n}$, which is the Gauss image of W^1_{n+2}. The family

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of all the n-spaces $\langle D\rangle$, as L varies in W^1_{n+2}, is 2-dimensional, i.e. a congruence. On each sufficiently general $\langle D\rangle$ belonging to the congruence we consider the locus of first order foci and we prove that it is a rational normal curve containing $\mathrm{Supp}(D_s)\cup\{v_L\}$ (theorem 2).

As D varies the first order foci fill a 3-dimensional irreducible variety F_C containing $\kappa(C)\cup\Gamma$. This variety is also the union of the 1-dimensional family of surfaces F_L swept by the focal curves as D varies in a linear pencil $|L|$, $L\in W^1_{n+2}$. We completely explain the geometrical meaning of these surfaces.

We then consider the second order foci of the above family of n-spaces, proving that on every sufficiently general $\langle D\rangle$ they are the $n+3$ points $\big(\kappa(C)\cup\Gamma\big)\cap\langle D\rangle$ (theorem 3). This allows to reconstruct the curve C from the congruence. In the final section we consider the genus 5 case, proving that F_C is a hypersurface of degree 40 in $\mathbb{P}^4$ (theorem 5).

1. First order foci

For a fixed integer $n\geq 2$ let C be a projective irreducible nonsingular and non hyperelliptic curve of genus $g=2n+1$ and let $\mathbb{P}=\mathbb{P}(V)\cong\mathbb{P}^{2n}$, where $V=H^1(C,\mathcal{O}_C)$. Denote by

$$\kappa: C\to\mathbb{P}$$

the canonical embedding of C, defined by the complete canonical linear series $|K|$.

As customary, we will use the symbol g^r_d to mean "a linear series of dimension r and degree d".

We denote by C_d the d-th symmetric product of C and by $W_d(C)$ the image of the Abel-Jacobi map

$$\alpha_d: C_d\to\mathrm{Pic}^d(C)$$

Moreover we let

$$C^r_d=\{D\in C_d: h^0(D)\geq r+1\}$$

and

$$W^r_d=W^r_d(C)=\alpha_d(C^r_d)=\{L\in\mathrm{Pic}^d(C): h^0(L)\geq r+1\}$$

We will consider C^r_d and W^r_d with their natural scheme structure.

In particular we will consider W^1_{n+2}. By Brill-Noether theory $W^1_{n+2}\neq\emptyset$ and all its components have dimension at least one. Moreover, if $C\in\mathcal{M}_g$ is sufficiently general then W^1_{n+2} is an irreducible and nonsingular curve.

LEMMA 1 *If C is a general curve of genus $g=2n+1$ then for every $L\in W^1_{n+2}$ the linear series $|L|$ is a base point free pencil, $H^1(C,L^2)=0$ and $|L^2|$ is a g^3_{g+3} not composed with an involution.*

Proof

If $|L|$ has a base point p then $L(-p)\in W^1_{n+1}$, and this contradicts the generality of C. Similarly $|L|$ cannot have dimension bigger than one without contradicting the generality of C.

$H^1(C,L^2)=0$ follows from $H^0(C,KL^{-2})\cong\ker(\mu_L)$, where

$$\mu_L: H^0(C,L)\otimes H^0(C,KL^{-1})\to H^0(C,K)$$

is the Petri map, and since this map is injective by the generality of C (see [G]).

By Riemann-Roch $h^0(C,L^2)=4$. Then $\varphi_{L^2}: C\to\mathbb{P}^3$ is not composed with an irrational involution by the generality of C. Similarly φ_{L^2} is not composed with a rational involution because by the generality of C there is no g^1_h with $h < n+2$, a contradiction. *q.e.d.*

We will henceforth assume that C is a general curve of genus g, so that W^1_{n+2} has the above stated properties. It follows that $\alpha_{n+2} : C^1_{n+2} \to W^1_{n+2}$ is a $\mathbb{P}^1$-bundle, and in particular C^1_{n+2} is an irreducible nonsingular surface, which we will denote by S.

For every $s \in S$ we let D_s be the divisor of degree $n+2$ parametrized by s and $\Lambda_s = \langle D_s \rangle \subset \mathbb{P}$ its linear span, which is a $\mathbb{P}^n$, $(n+2)$-secant to the curve $\kappa(C)$. We therefore have a two-dimensional family of $(n+2)$-secant $\mathbb{P}^n$'s parametrized by S:

$$\begin{array}{lll} \underline{\Lambda} & \subset & S \times \mathbb{P} \\ \downarrow \pi & & \\ S & & \end{array} \tag{1}$$

Let's recall how this family is constructed. Consider the universal divisor of degree $n+2$:

$$\mathbf{D}_{n+2} \subset C_{n+2} \times C$$

and let $\mathbf{D}_S = \mathbf{D}_{n+2} \cap (S \times C)$. Denote by $p : S \times C \to S$ the projection. We have a homomorphism of locally free sheaves on S:

$$R^1 p_* \mathcal{O}_{S\times C} \to R^1 p_* \mathcal{O}_{S\times C}(\mathbf{D}_S) \to 0$$

whose kernel is a locally free subsheaf $\mathcal{F} \subset R^1 p_* \mathcal{O}_{S\times C} \cong H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_S$ of rank $n+1$. Taking the associated projective bundles we get

$$\underline{\Lambda} = \mathbb{P}(\mathcal{F}) \subset \mathbb{P}(R^1 p_* \mathcal{O}_{S\times C}) = S \times \mathbb{P}$$

For each point $L \in W^1_{n+2}$ we have the 1-dimensional subfamily of (1) parametrized by the fibre $\alpha^{-1}(L)$, i.e. consisting of the $\Lambda_s = \langle D_s \rangle$ when D_s varies in the linear pencil $|L|$. Their union $X_L := \cup_{s \in \alpha^{-1}(L)} \Lambda_s$ is an irreducible variety of dimension $n+1$ and degree n (a rational scroll of minimal degree) containing the canonical curve $\kappa(C)$. It can be constructed as follows.
Let $\{\sigma_1, \sigma_2\}$ be a basis of $H^0(C, L)$ and $\{\tau_1, \ldots, \tau_n\}$ a basis of $H^0(C, KL^{-1})$. Then X_L is the variety defined by the $\binom{n}{2}$ quadrics obtained as maximal minors of the matrix of linear forms:

$$\begin{pmatrix} Z_{11} & Z_{12} & \cdots & Z_{1n} \\ Z_{21} & Z_{22} & \cdots & Z_{2n} \end{pmatrix}$$

where $Z_{ij} = \sigma_i \tau_j$. Since the Brill-Noether map

$$\mu_L : H^0(C, L) \otimes H^0(C, KL^{-1}) \to H^0(C, K)$$

is injective, it follows that the Z_{ij}'s are linearly independent and therefore X_L is a cone with vertex a point v_L. A theorem of Kempf [K1] describes X_L as the projectivized tangent cone to W_{n+2} at L.
Note that in the case $g = 5$ the variety X_L is a quadric of rank 4 containing the canonical curve.

As L varies in W^1_{n+2} we obtain a 1-dimensional family of $(n+1)$-dimensional cones X_L whose vertices describe a curve $\Gamma \subset \mathbb{P}$. This curve is the Gauss image of W^1_{n+2}, because for each $L \in W^1_{n+2}$ the vertex v_L is the projectivized tangent line to W^1_{n+2} at L. We have the obvious equality:

$$\bigcup_{L \in W^1_{n+2}} X_L = p_2(\underline{\Lambda})$$

where $p_2 : \underline{\Lambda} \to \mathbb{P}$ is the projection.

Let $\Lambda = \Lambda_s$, $s \in S$, be a member of the family (1). We have $\Lambda = \mathbb{P}(U)$ where $U \subset V$ is a vector subspace of dimension $n+1$. The normal bundle of Λ in $\mathbb{P}$ is

$$N_\Lambda = \frac{V}{U} \otimes_{\mathbf{k}} \mathcal{O}_\Lambda(1)$$

and

$$H^0(\Lambda, N_\Lambda) = \mathrm{Hom}(U, \frac{V}{U})$$

Associated to the family (1) we have a functorial morphism $\chi : S \to G_n(\mathbb{P})$ whose differential at s is a linear map (the characteristic map)

$$d\chi_s : T_{S,s} \to H^0(\Lambda, N_\Lambda)$$

The characteristic map induces a homomorphism of locally free sheaves on Λ:

$$\Phi_s : T_{S,s} \otimes_{\mathbf{k}} \mathcal{O}_\Lambda \to N_\Lambda$$

We define *the scheme of first order foci* (or the *focal scheme*) of the family (1) at s to be

$$F_s = D_1(\Phi_s) = \{x \in \Lambda_s : \mathrm{rk}_x(\Phi_s) \leq 1\}$$

Its points are called *first order foci* of the family (1) at s.

THEOREM 2 *For every $L \in W^1_{n+2}$ and for all sufficiently general $s \in \alpha^{-1}_{n+2}(L)$, F_s is a rational normal curve (of degree n) of Λ_s containing* $\mathrm{Supp}(D_s) \cup \{v_L\}$.

Proof

Since $\dim(T_{S,s}) = 2$, Φ_s is defined by a $n \times 2$ matrix of linear forms on Λ. Therefore, to show that F_s is a rational normal curve it will be sufficient to show that Φ_s is 1-generic (see [E], theorem 2.1 and section 2 of [CS]). This means that for every tangent vector $\theta \in T_{S,s} \backslash \{0\}$ the element $d\chi_s(\theta) \in H^0(\Lambda, N_\Lambda)$, interpreted as a homomorphism $d\chi_s(\theta) : U \to \frac{V}{U}$, is surjective.

Consider the first order deformation of Λ_s defined by $d\chi_s(\theta)$:

$$\begin{array}{ccc} \underline{\Lambda}_\epsilon & \subset & \mathrm{Spec}(\mathbf{k}[\epsilon]) \times \mathbb{P} \\ \downarrow & & \\ \mathrm{Spec}(\mathbf{k}[\epsilon]) & & \end{array}$$

The surjectivity of $d\chi_s(\theta)$ is equivalent to the fact that $p_2(\underline{\Lambda}_\epsilon) \subset \mathbb{P}$ is not contained in a hyperplane. But

$$p_2(\underline{\Lambda}_\epsilon) \supset p_2(\mathbf{D}_\epsilon)$$

where $\mathbf{D}_\epsilon \subset \mathrm{Spec}(\mathbf{k}[\epsilon]) \times \mathbb{P}$ is the first order deformation of the divisor D_s defined by θ. Note that $p_2(\mathbf{D}_\epsilon)$ is a curvilinear scheme corresponding to a divisor on C satisfying:

$$D_s \leq p_2(\mathbf{D}_\epsilon) \leq 2D_s$$

Assume first that θ is tangent to $\alpha^{-1}_{n+2}(L)$ at s, equivalently that the family $\mathbf{D}_\epsilon$ deforms D_s in the linear pencil $|L|$. Then, letting $\varphi_L : C \to \mathbb{P}^1$ be the morphism defined by the pencil, we have $p_2(\mathbf{D}_\epsilon) = \varphi^*_L(\theta)$, where we have identified θ with a curvilinear subscheme of $\mathbb{P}^1$ supported at the point $s \in \mathbb{P}^1$. Since $|D_s| = |L|$ is base point free, it follows that $p_2(\mathbf{D}_\epsilon) = 2D_s$, and therefore $p_2(\underline{\Lambda}_\epsilon)$ is contained in a hyperplane if and only if $2D_s$ is a special divisor. But this is not possible by lemma 1; hence $d\chi_s(\theta)$ is surjective in this case.

Assume now that $\theta \in T_{S,s}\backslash\{0\}$ is not tangent to $\alpha_{n+2}^{-1}(L)$ at s. We can write:

$$p_2(\mathbf{D}_\epsilon) = p_1 + \cdots + p_k + 2(p_{k+1} + \cdots + p_{n+2})$$

where $D_s = p_1 + \cdots + p_{n+2}$, and $k \geq 0$.
If $k = 0$ we conclude as in the previous case that $d\chi_s(\theta)$ is surjective.
If $k = 1$ then $p_2(\mathbf{D}_\epsilon) = p_1 + 2(p_2 + \cdots + p_{n+2}) = 2D_s - p_1$. If this is a special divisor then, since $2D_s$ is non special, the linear series $|2D_s|$ has a base point, which is absurd. Therefore $2D_s - p_1$ is non special, i.e. $p_2(\mathbf{D}_\epsilon)$ is not contained in a hyperplane, and again $d\chi_s(\theta)$ is surjective.
Assume that $k \geq 2$: we will show that this case cannot occur. The vector θ is also tangent to $p_1 + \cdots + p_k + C_{(n+2-k)}$. Consider the differential of α_{n+2} at s, which is identified with the linear map:

$$H^0(D_s, \mathcal{O}_{D_s}(D_s)) \xrightarrow{\delta} H^1(C, \mathcal{O}_C)$$

arising from the exact sequence

$$0 \to \mathcal{O}_C \to L \to \mathcal{O}_{D_s}(D_s) \to 0$$

In $H^0(D_s, \mathcal{O}_{D_s}(D_s)) = T_{C_{n+2},s}$ the tangent subspace to $p_1 + \cdots + p_k + C_{(n+2-k)}$ is the subspace $H^0(E_s, \mathcal{O}_{E_s}(D_s))$, where we have denoted $E_s = p_{k+1} + \cdots + p_{n+2}$. Therefore $\theta \in H^0(E_s, \mathcal{O}_{E_s}(D_s))$. Applying δ and then projectivizing we deduce that

$$[\delta(\theta)] \in \langle E_s \rangle = \langle p_{k+1} + \cdots + p_{n+2} \rangle \subset \Lambda_s \subset I\!P$$

Since θ is not tangent to $\alpha_{n+2}^{-1}(L)$, we have that $[\delta(\theta)] = v_L$,the vertex of the cone X_L, and therefore we see that for every sufficiently general $D \in |L|$ there is an effective divisor E of degree n such that $D = E + p + q$ and $v_L \in \langle E \rangle$.
This means that $\dim(\langle D_s + E \rangle) = 2n - 1$, equivalently:

$$3 = h^0(C, \mathcal{O}(D_s + E)) = h^0(C, \mathcal{O}(L^2(-p - q))$$

for infinitely many $p + q$. Since $h^0(C, \mathcal{O}(L^2)) = 4$ this implies that $|L^2|$ is composed with an involution, and this contradicts lemma 1. This proves that F_s is a rational normal curve.

Finally note that for every $p \in \mathrm{Supp}(D_s) \cup \{v_L\}$ the space Λ_s belongs to a 1-dimensional subfamily of (1) consisting of $I\!P^n$'s containing p. Correspondingly there is a nonzero tangent vector $\theta_p \in T_{S,s}$ such that $d\chi_s(\theta_p) : U \to \frac{V}{U}$ vanishes at p. This implies that $p \in F_s$. *q.e.d.*

Note that as a consequence of theorem 2 we have that, for each sufficiently general $s \in S$, $\mathrm{Supp}(D_s) \cup \{v_L\}$ consists of $n + 3$ distinct points in linearly general position in Λ_s, because they lie on the rational normal curve F_s. The focal curve F_s is actually the only rational normal curve of Λ_s containing the $n + 3$ distinct points $\mathrm{Supp}(D_s) \cup \{v_L\}$, and therefore it could be also defined as such.

For each $L \in W_{n+2}^1$ denote by $\mathcal{U} \subset \alpha_{n+2}^{-1}(L)$ the open subset such that $F_s \subset \Lambda_s$ is a rational normal curve. Define:

$$F_L = \overline{\bigcup_{s \in \mathcal{U}} F_s}$$

From theorem 2 it follows that F_L is an irreducible surface such that

$$\kappa(C) \subset F_L \subset X_L$$

The surface F_L can be also described as follows. By lemma 1 the morphism $\varphi_{L^2} : C \to I\!P^3$ maps C birationally onto a curve $\varphi_{L^2}(C)$ of degree $2n + 4 = g + 3$ which is contained in a quadric cone V, whose generating lines cut on C the pencil $|L|$. Let $\sigma : U \to V$ be the blow up of V at the conductor ideal of $\varphi_{L^2}(C)$, and $C' \subset U$ the proper transform of $\varphi_{L^2}(C)$. Then the adjoint morphism $\varphi_{K_U+C'} : U \to I\!P$ maps U onto a surface containing $\kappa(C)$ which contains a 1-parameter family of rational curves of degree n (the images of the proper transforms of the lines of V) which cut on $\kappa(C)$ the linear series $|L|$. It follows that these curves are the focal curves and $\varphi_{K_U+C'}(U) = F_L$.

We define:

$$F_C = \bigcup_{L \in W^1_{n+2}} F_L$$

Note that $F_L \neq F_C$ for each $L \in W^1_{n+2}$ because otherwise $F_L = F_{L'}$ for all $L' \in W^1_{n+2}$, and the surface F_L would contain the 2-dimensional family of focal curves. But these curves would then be linearly equivalent on the desingularization U of F_L, and therefore they would define a g^2_{n+2} on C, which is impossible (lemma 1).

Therefore $F_C \subset I\!P$ is a 3-dimensional irreducible variety containing $\kappa(C) \cup \Gamma$, and intrinsically defined by C. We will call it *the variety of first order foci of* W^1_{n+2}.

When $g = 5$ F_C is a hypersurface of $I\!P^4$ which we will consider again in §3, where we will compute its degree.

2. Second order foci

We have seen that the family (1) defines, for each $s \in S$, a closed subscheme $F_s \subset \Lambda_s$ of first order foci. All these subschemes fit together in a closed subscheme $\mathcal{F} \subset \underline{\Lambda}$ which is defined as

$$\mathcal{F} = D_1(\Phi)$$

where

$$\Phi : \pi^*(T_S) \to N_{\underline{\Lambda}/(S \times I\!P)}$$

is the characteristic homomorphism. We obtain a morphism $\pi_1 : \mathcal{F} \to S$ and a diagram:

$$\begin{array}{lll} \mathcal{F} & \subset & S \times I\!P \\ \downarrow \pi_1 & & \\ S & & \end{array} \tag{2}$$

From theorem 2 it follows that for all sufficiently general $s \in S$ the fibre $F_s = \pi_1^{-1}(s)$ is a rational normal curve of degree n in Λ_s. For such an s we can introduce the *second order foci* of the family (1), defined as the first order foci of the family (2) at s. Namely, we consider the homomorphism

$$\xi_s : T_{S,s} \otimes \mathcal{O}_{F_s} \to N_{F_s}$$

where N_{F_s} is the normal bundle of F_s in $I\!P$, induced by the characteristic map

$$d\chi_{1s} : T_{S,s} \to H^0(F_s, N_{F_s})$$

(which is the differential of the functorial morphism $\chi_1 : S^\circ \to Hilb^{I\!P}$ defined on an open neighborhood S° of s in S); then we define *the scheme of second order foci of the family* (1) *at* s as

$$D_1(\xi_s) \subset F_s$$

i.e. as the closed subscheme of F_s defined by the condition $\mathrm{rk}(\xi_s) \leq 1$.

THEOREM 3 *Let $s \in S$ be a sufficiently general point. Then we have:*

$$D_1(\xi_s) = \Lambda_s \cap (\kappa(C) \cup \Gamma)$$

or, equivalently:

$$D_1(\xi_s) = \mathrm{Supp}(D_s) \cup \{v_L\}$$

where $L = \mathcal{O}(D_s)$. In particular the scheme of second order foci of the family (1) at s is a zero-dimensional closed subscheme of degree $n+3$ of F_s.

Proof

Let $p \in \mathrm{Supp}(D_s) \cup \{v_L\}$. Then, as remarked in the proof of theorem 2, there is an irreducible curve $B \subset S$ containing s such that $p \in \Lambda_{s'}$ for each $s' \in B$. It follows that $p \in F_{s'}$ for each $s' \in B$ and therefore there is a nonzero tangent vector $\theta_p \in T_{S,s}$ such that $d\chi_{1s}(\theta_p) \in H^0(F_s, N_{F_s})$ vanishes at p. Therefore $p \in D_1(\xi_s)$ and we have an inclusion

$$\mathrm{Supp}(D_s) \cup \{v_L\} \subset D_1(\xi_s)$$

Since $\mathrm{Supp}(D_s) \cup \{v_L\}$ consists of $n+3$ distinct points, by the generality of s, the theorem will follow if we prove that $\deg(D_1(\xi_s)) \leq n+3$.

Let's denote by ℓ the line bundle of degree 1 on F_s. We have a direct sum decomposition:

$$N_{F_s} \cong \left[\oplus^n \ell^n \right] \oplus \left[\oplus^{n-1} \ell^{n+2} \right]$$

corresponding to the decomposition:

$$N_{F_s} \cong (N_{\Lambda_s} \otimes \mathcal{O}_{F_s}) \oplus N_{F_s/\Lambda_s}$$

After a choice of a basis of $T_{S,s}$ and of these decompositions the homomorphism ξ_s is represented by a matrix of the form:

$$M = \begin{pmatrix} A \\ B \end{pmatrix}$$

where A is a $n \times 2$ matrix of sections of ℓ^n, and B is a $(n-1) \times 2$ matrix of sections of ℓ^{n+2}.

Note that A is the restriction to F_s of the matrix representing the homomorphism $\Phi_s : T_{S,s} \otimes \mathcal{O}_\Lambda \to N_\Lambda$; in particular A has rank one. More precisely choices can be made so that

$$A = \begin{pmatrix} t^n & st^{n-1} \\ st^{n-1} & s^2t^{n-2} \\ \dots & \dots \\ s^{n-1}t & s^n \end{pmatrix}$$

where $\{t, s\}$ is a basis of $H^0(F_s, \ell)$. From this expression we see that the i-th row of A is

$$t^{n-i}s^{i-1} \begin{pmatrix} t & s \end{pmatrix}$$

It follows that

$$D_1(\xi_s) = D_1(N)$$

where

$$N = \begin{pmatrix} t & s \\ & B \end{pmatrix}$$

Since the entries of B are sections of ℓ^{n+2}, we deduce that $\deg(D_1(\xi_s)) \leq n+3$, and the conclusion follows. *q.e.d.*

The following corollary is now immediate:

COROLLARY 4 *The family (1) uniquely determines $\kappa(C) \cup \Gamma$ as the closure of the union of its second order foci. In particular (1) uniquely determines $\kappa(C)$.*

3. The genus 5 case

In this section we will assume that C is a general curve of genus $g = 5$ and we will compute the degree of the focal variety F_C. In this case F_C is a hypersurface of $I\!P \cong I\!P^4$ which can be also described in the following way.

Let Σ be the net of quadrics of $I\!P^4$ containing $\kappa(C)$ and let $\Delta \subset \Sigma$ be the discriminant curve, i.e. the locus parametrizing singular quadrics. By the generality of C we have that Δ is a nonsingular quintic curve which parametrizes the rank 4 quadrics Q_L, $L \in W^1_4$, and the congruence (1) consists of all the planes contained in the quadrics Q_L.

For each $p \in I\!P^4 \backslash \kappa(C)$ denote by $\Sigma_p \subset \Sigma$ the pencil of quadrics of Σ containing p. Since the curve Γ is the locus of vertices of the quadrics Q_L, for each $v_L \in \Gamma$ the pencil Σ_{v_L} is the tangent line to Δ at the point $[Q_L]$.

On a plane $\Lambda \subset Q_L$ each quadric $Q \in \Sigma_{v_L}$ different from Q_L cuts the focal conic F, because $\Lambda \cap Q$ is a conic containing $\{v_L\} \cup (\kappa(C) \cap \Lambda)$; therefore $\Sigma_p = \Sigma_{v_L}$ for each $p \in F$. We therefore conclude that F_C *is the closure of the set of points $p \in I\!P^4 \backslash \kappa(C)$ such that $\Sigma_p = \Sigma_{v_L}$ for some $L \in W^1_4$*. Equivalently, denoting by $\Delta^* \subset \Sigma^*$ the dual curve of Δ:

$$F_C = \overline{\{p \in I\!P^4 \backslash \kappa(C) : \Sigma_p \in \Delta^*\}}$$

Let's consider the rational map defined by Σ:

$$\varphi_\Sigma : I\!P^4 - -- \to \Sigma^*$$

Then

$$F_C = \overline{\varphi_\Sigma^{-1}(\Delta^*)}$$

Therefore, if λ is a general line of $I\!P^4$, we have:

$$\deg(F_C) = \deg(\lambda \cap F_C) = \deg(\varphi_\Sigma(\lambda) \cap \Delta^*) = 20 \deg(\varphi_\Sigma(\lambda)) \tag{3}$$

because $\deg(\Delta^*) = 20$.

If $t \subset \Sigma^*$ is a general line, $\overline{\varphi_\Sigma^{-1}(t)}$ is a general quadric of the net Σ. It follows that

$$\deg(\varphi_\Sigma(\lambda)) = \deg(t \cap \varphi_\Sigma(\lambda)) = \deg(\overline{\varphi_\Sigma^{-1}(t)} \cap \lambda) = 2$$

Comparing with (3) we deduce that $\deg(F_C) = 40$. Therefore we have proved that following:

THEOREM 5 *If C is a general curve of genus 5, the focal hypersurface $F_C \subset I\!P^4$ has degree 40.*

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